Games, graphs, and machines

matrix (n.)

late 14c., *matris*, *matrice*, "uterus, womb," from Old French *matrice* "womb, uterus" and directly from Latin *mātrix* (genitive *mātricis*) "pregnant animal," in Late Latin "womb," also "source, origin," from *māter* (genitive *mātris*) "mother" (see **mother** (n.1)).

August 21, 2024

Warm up

Remember that $\oplus = \min$ and $\odot = +$. Find

Same oop as with usual $\times$ and $+$.

$$\underbrace{2 \odot \underbrace{(3 \oplus 1)}_{1}}_{3} \oplus \underbrace{3 \odot \underbrace{(\infty \oplus 2)}_{2}}_{5}. = 3$$

$$\underbrace{3 \oplus 5}_{3}$$

$a + \min(b, c)$

$a \odot (b \oplus c)$

$= a \odot b \oplus a \odot c$

$\min(a+b, a+c)$

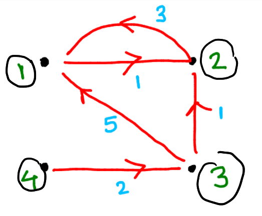
$\oplus$ = cost of "or"

$\odot$ = cost of "and"

Weighted adjacency matrix

Write the min/plus weighted adjacency matrix of the graph. Assume that the loops have weight 0 (not shown).

$$a \oplus \infty = a$$



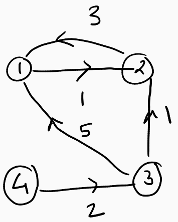
	1	2	3	4
1	0	1	∞	∞
2	3	0	∞	∞
3	5	1	0	∞
4	∞	∞	2	0

Min/plus powers

Find the min/plus square and cube of the adjacency matrix. What do its entries represent?

$$\begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 5 & 1 & 0 & \infty \\ \infty & \infty & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 5 & 1 & 0 & \infty \\ \infty & \infty & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 4 & 1 & 0 & \infty \\ 7 & 3 & 2 & 0 \end{pmatrix} = A^2$$

$$A^{\odot 3} = \begin{pmatrix} 0 & 1 & \infty & \infty \\ 3 & 0 & \infty & \infty \\ 4 & 1 & 0 & \infty \\ 6 & 3 & 2 & 0 \end{pmatrix}$$



$A^{\odot n}$ entries are cheapest costs of length n paths

Why do min/plus powers give shortest paths?

For example, the third power:

$$
\begin{aligned}
A^3_{i,j} &= (A^2_{i,1} \odot A_{1,j}) \oplus (A^2_{i,2} \odot A_{2,j}) \oplus (A^3_{i,3} \odot A_{3,j}) \oplus (A^2_{i,4} \odot A_{4,j}) \\
&= \min(A^2_{i,1} + A_{1,j}, A^2_{i,2} + A_{2,j}, A^2_{i,3} + A_{3,j}, A^2_{i,4} + A_{4,j})
\end{aligned}
$$

Length 2 from i to 1 <u>and</u> length 1 from 1 to j

When do we stop?

Assume:

1. we have all loops with weight 0,
2. all weights are non-negative.

Theorem
Let n be the number of vertices. Then $A^{\odot(n-1)} = A^{\odot n} = A^{\odot(n+1)} = \cdots$.

Why? A length $(n-1)$ path is the longest without cycles.

Cheapest of length $(n-1)$ = cheapest of len up to $(n-1)$.

Cheapest path can be found among len up to $(n-1)$ paths.